Notes on homology theory

① Homology as a functor :

let k be a fixed field
(typically, $k = \mathbb{Z}/p\mathbb{Z}$ for some prime p in practice)

$$H_*(\,\cdot\,; k) : \quad \mathrm{Top} \longrightarrow \mathrm{Vect}_k$$

topo. space $\qquad$ k-vector space

$$X \longmapsto H_*(X; k)$$

$$(X \xrightarrow{f} Y) \longmapsto H_*(X;k) \xrightarrow{H_*(f;k)} H_*(Y;k)$$

continuous map $\qquad$ k-linear map

such that :

$$\forall X, \quad H_*(\mathrm{id}_X; k) = \mathrm{id}_{H_*(X;k)}$$

$$\forall X \xrightarrow{f} Y \xrightarrow{g} Z, \quad H_*(g \circ f; k) = H_*(g;k) \circ H_*(f;k)$$

$\rightsquigarrow$ in diagram format :

$$\begin{tikzcd} X \arrow[loop right, "\mathrm{id}_X"] \end{tikzcd} \longmapsto \begin{tikzcd} H_*(X;k) \arrow[loop right, "\mathrm{id}_{H_*(X;k)}"] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & \circlearrowleft & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd} \longmapsto \begin{tikzcd} H_*(X;k) \arrow[rr, "H_*(g\circ f;k)"] \arrow[dr, "H_*(f;k)"'] & & H_*(Z;k) \\ & H_*(Y;k) \arrow[ur, "H_*(g;k)"'] & \end{tikzcd}$$

Invariance properties:

Prop: $X \underset{\text{(homeo.)}}{\simeq} Y \Rightarrow H_*(X;k) \underset{\text{(iso)}}{\simeq} H_*(Y;k)$.

→ **proof:** let $f: X \to Y$ homeo. Then:

$$\begin{cases} id_{H_*(X;k)} = H_*(id_X;k) = H_*(f^{-1} \circ f;k) = H_*(f^{-1};k) \circ H_*(f;k) \\ id_{H_*(Y;k)} = H_*(id_Y;k) = H_*(f \circ f^{-1};k) = H_*(f;k) \circ H_*(f^{-1};k). \end{cases}$$

Hence, $H_*(f;k)$ is an iso. $H_*(X;k) \to H_*(Y;k)$. ∎

Prop: $X \underset{\text{(homotopy equiv.)}}{\simeq} Y \Rightarrow H_*(X;k) \underset{\text{(iso.)}}{\simeq} H_*(Y;k)$.

→ **proof:** same as above, using the following fact:

Prop: $f \underset{\text{(homotopic)}}{\sim} g : X \to Y \Rightarrow H_*(f;k) = H_*(g;k)$.

∎

(Reminder):

Let X, Y be two topological spaces.

$\boxed{\text{Def:}}$ Homeomorphism: $h: X \to Y$ bijective s.t. both h and h^{-1} are continuous.
↳ then X, Y are homeomorphic.

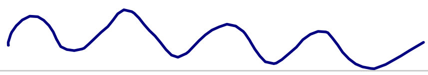
$\boxed{\text{Def:}}$ $f, g: X \to Y$ are homotopic (noted $f \sim g$) if
$\exists\, H: [0,1] \times X \to Y$ continuous s.t.
$H(0,\cdot) = f$ and $H(1,\cdot) = g$.
↳ H is called a homotopy.

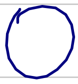
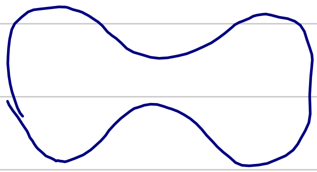
$\boxed{\text{Def:}}$ $f: X \to Y$ is a homotopy equivalence if
$\exists\, g: Y \to X$ s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$.
↳ X, Y are said homotopy equivalent.

$\boxed{\text{Prop:}}$ X homeo. to $Y \Rightarrow X$ homotopy equiv. to Y.
→ proof: given $h: X \to Y$ homeo., take
$f := h$ and $g := h^{-1}$.

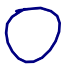
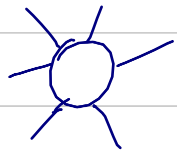
$\Rightarrow g \circ f = id_X \sim id_X$ and $f \circ g = id_Y \sim id_Y$. $\blacksquare$

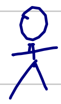
Examples:

 homeo. to 

 homeo. to 

 homotopy equiv. to 

 homotopy equiv. to 

and to 

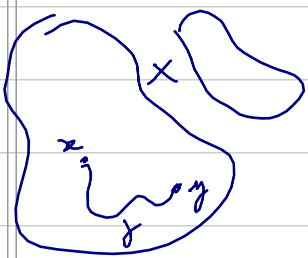
Note: These notions are introduced for <u>classifying topological spaces</u> (up to homeo., homotopy equiv., etc.)

↳ homotopy theory is a <u>tool for classification</u>.

② Intuition of homology:

$\boxed{\text{Def.}}$ A path is a continuous map $[0,1] \to X$.

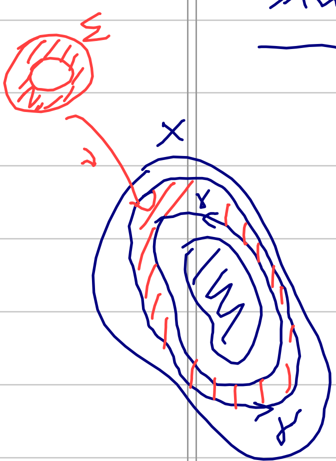
$\boxed{\text{Def:}}$ $x \sim y$ in X iff $\exists$ path $\gamma : [0,1] \to X$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$. (rmk: equivalence relation)



$\boxed{\text{Def:}}$ (Path-) connected components of $X \equiv X/\sim$ (equivalence classes of the relation $\sim$).

Alternative formulation: $x \sim y$ in X if $\exists$ path γ s.t. $\gamma(\underbrace{\partial[0,1]}_{=\{0,1\}}) = \{x, y\}$

Similarly: - A loop is a map $\gamma : S^1 \to X$ that is continuous.



- Two loops $\gamma, \gamma' : S^1 \to X$ are equivalent if $\exists$ surface Σ and a map $\partial : \Sigma \to X$ s.t. $\partial(\partial \Sigma) = \gamma(S^1) \cup \gamma'(S^1)$.

↳ higher-dimensional generalizations:

- n-cycle $\equiv$ map $\gamma : \Sigma \to X$ where $\dim_{\mathbb{R}} \Sigma = n$ and $\partial \Sigma = \emptyset$
- $\gamma \sim \gamma'$ if $\exists\ \partial : \Omega \to X$ s.t. $\partial(\partial \Omega) = \gamma(\Sigma) \cup \gamma'(\Sigma')$, where $\dim_{\mathbb{R}} \Omega = n+1$

↳ leads to $\boxed{\text{cobordism theory}}$ → homology simplifies it via linear algebra

Idea: build cycles by gluing basic building blocks (simplices).

↳ use a discretization of the space.

③ Simplicial complexes: (reminder)

Def: A graph is given by:
- a vertex set V
- an edge set $E \subseteq \underbrace{\mathcal{P}_2(V)}_{\text{set of pairs } \{v_i, v_j\}. \ (i \neq j)}$

Note: $E \subseteq \mathcal{P}_2(V) \Rightarrow$ every edge has its 2 vertices in the vertex set

↳ direct generalization:

Def: A simplicial complex is given by:
- a vertex set V
- a simplex set $K \subseteq \underbrace{\mathcal{P}(V)}_{\text{all subsets of } V}$ such that:
  - $\emptyset \notin K$
  - $\forall \sigma \in K, \forall \underset{\emptyset \neq}{\tau} \subseteq \sigma, \tau \in K.$

  subsets of vertices.

↳ Glossary:
- the complex is often denoted K (vertex set is implicit, assimilated to sets of size 1 in K)
- every $\sigma \in K$ is a simplex
  - ↳ vertex if $\#\sigma = 1$
  - ↳ edge if $\#\sigma = 2$

$\#\cdot$: cardinality

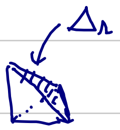
↳ triangle if $\#\sigma = 3$
↳ tetrahedron if $\#\sigma = 4$

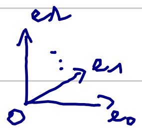
...

- $\tau \leq \sigma \rightsquigarrow \tau$ is a face of σ.
- $\dim \sigma := \#\sigma - 1$ || $\dim K = \sup_{\sigma \in K} \dim \sigma$.
- $K_n = \{\sigma \in K \mid \dim \sigma = n\}$ (k-skeleton)

$\boxed{Q}$ How to turn it into a topological space?

$\boxed{A}$ By drawing it ;-)





Def. The standard n-simplex Δ_n is defined by:

$$\Delta_n := \mathrm{Conv}\{e_0, e_1, \dots, e_n\} \text{ in } \mathbb{R}^{n+1}$$
$$= \left\{ \sum_{i=0}^{n} \lambda_i e_i \;\middle|\; \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

Def: A geometric realization of a simplicial complex K is a topological space X together with a family of continuous maps $(f_\sigma : \Delta_{\dim \sigma} \to X)_{\sigma \in K}$ such that:

- $X = \bigcup_{\sigma \in K} \mathrm{Im}\, f_\sigma$;
- each f_σ is injective;
- $\forall \sigma, \tau \in K$, $\mathrm{Im}\, f_\sigma \cap \mathrm{Im}\, f_\tau = \mathrm{Im}\, f_{\sigma \cap \tau}$ (with the convention that $\mathrm{Im}\, f_\emptyset = \emptyset$)
- $U \subseteq X$ is open in X iff $f_\sigma^{-1}(U)$ is open in $\Delta_{\dim \sigma}$ for all $\sigma \in K$.

Prop: All geometric realizations of K are homeomorphic.

↳ **Def:** The underlying space $|K|$ is any geometric realization of K.

Def: A space X is *triangulable* if $\exists K$ such that X and $|K|$ are homeomorphic.
↳ K is then a *triangulation* of X.

4) Simplicial homology:

Let k be a fixed field.
(here we assume $k = \mathbb{Z}/2\mathbb{Z}$ for simplicity).

4.1 Chains:

Def: n-chain = formal k-weighted sum of n-simplices
$\equiv$ map $K_n \to k$ with finite support.

$$c : K_n \to k \qquad\qquad c := \sum_{\sigma \in K_n} \underset{\in k}{c(\sigma)}\ \underset{\in K_n}{\sigma}$$
$$\sigma \mapsto c(\sigma)$$

s.t. $c(\sigma) = 0$ except for finitely many σ.

Note: maps $K_n \to k$ with finite support form a k-vector space:

still finite support:
$$(c+c')(\sigma) = c(\sigma) + c'(\sigma) \quad \text{(sum in } k\text{)}$$
$$(\lambda c)(\sigma) = \lambda\, c(\sigma) \quad \text{(product in } k\text{)}$$

→ call $C_n(K)$ this vector space ("space of chains")

4.2 Boundary Operator:

Hyp: $k = \mathbb{Z}/2\mathbb{Z}$ (characteristic $= 2$)

Def: $\partial_n : C_n(K) \to C_{n-1}(K)$

(def. on basis elements)
$$\sigma = \{v_0, \dots, v_n\} \longmapsto \sum_{i=0}^{n} \underbrace{\{v_0, \dots, \hat{v_i}, \dots, v_n\}}_{\sigma \setminus \{v_i\}}$$
(removed from set)

(extend by linearity)
$$c = \sum_{\sigma \in K_n} c(\sigma)\sigma \longmapsto \sum_{\sigma \in K_n} c(\sigma)\, \partial\sigma.$$

Recall: want
"cycles" $\equiv$ zero boundary
"equiv. cycles" $\equiv$ difference is a boundary

$\boxed{\text{Prop:}}$ $\partial_{n-1} \circ \partial_n = 0 \quad \forall n \geq 1$.

<u>proof</u>: show it on basis elements, then will extend to the whole space $C_n(K)$ by linearity.

Let $\sigma = \{v_0, \dots, v_n\} \in C_n(K)$.

$$\partial_n(\sigma) = \sum_{i=0}^{n} \{v_0, \dots, \hat{v_i}, \dots, v_n\} = \sum_{i=0}^{n} \sigma \setminus \{v_i\}.$$

$$\partial_{n-1} \circ \partial_n (\sigma) = \sum_{i=0}^{n} \sum_{\substack{j=0 \\ j \neq i}}^{n} \sigma \setminus \{v_i, v_j\}$$

$$= \sum_{i=0}^{n} \left[\sum_{j<i} \{v_0, \dots, \hat{v_j}, \dots, \hat{v_i}, \dots, v_n\} + \sum_{j>i} \{v_0, \dots, \hat{v_i}, \dots, \hat{v_j}, \dots, v_n\} \right]$$

$$= \sum_{j<i} \{v_0, \dots, \hat{v_j}, \dots, \hat{v_i}, \dots, v_n\} + \sum_{j>i} \{v_0, \dots, \hat{v_i}, \dots, \hat{v_j}, \dots, v_n\}$$

$$= 2 \sum_{j>i} \{v_0, \dots, \hat{v_i}, \dots, \hat{v_j}, \dots, v_n\} = 0.$$

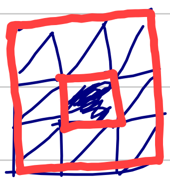
(characteristic $= 2$) ☐

<u>Hence</u>: $\underbrace{\operatorname{Im} \partial_n}_{\substack{(n-1)\text{-boundaries} \\ B_{n-1}(K)}} \subseteq \underbrace{\operatorname{Ker} \partial_{n-1}}_{\substack{(n-1)\text{-cycles} \\ Z_{n-1}(K)}}$.

$\boxed{\text{Def:}}$ $\boxed{H_n(K) := Z_n(K) / B_n(K)}$ n-th homology group.

<u>Note</u>: $c, c' \in Z_n(K)$ are equivalent ("homologous") iff $(c - c') \in B_n(K)$, i.e. $\exists\, c'' \in C_{n+1}(K)$ s.t. $(c - c') = \partial c''$.

<u>Example</u>:



The two red 1-cycles are homologous, and none is null-homologous.

⑤ Algorithms

Input: A finite simplicial complex K; a field k.
(here we assume $k = \mathbb{Z}/2\mathbb{Z}$ for simplicity)

Output: $H_r(K;k)$ $\forall r \geq 0$.
↪ $\simeq k^n$ for some $n \in \mathbb{N}$ → find n.

$$H_r := Z_r/B_r \Rightarrow \dim H_r = \underbrace{\dim Z_r}_{= \dim \ker \partial_r} - \underbrace{\dim B_r}_{= \operatorname{rank} \partial_{r+1}}$$

↳ $\forall r$ compute the matrix M_r of ∂_r in the simplex basis.

$$\begin{array}{c|c} & \sigma_1 \ \cdots \ \sigma_{\#K_r} \\ \hline \begin{array}{c} \tau_1 \\ \vdots \\ \tau_{\#K_{r-1}} \end{array} & \begin{bmatrix} & & \\ & M_r & \\ & & \end{bmatrix} \end{array} \rightsquigarrow \boxed{\dim H_r = \#K_r - \operatorname{rank} M_r - \operatorname{rank} M_{r+1}}$$

↳ $\forall r$ compute $\operatorname{rank} M_r$.

↳ possible approaches:

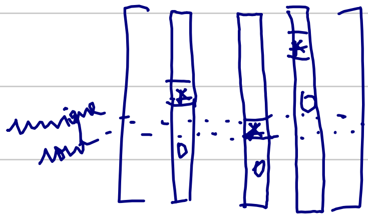
(i) Gaussian elimination $\rightsquigarrow O(m^3)$ where $m = \max\{\#K_r, \#K_{r-1}\}$.

(ii) Fast matrix multiplication $\rightsquigarrow O(m^\omega)$ where $\omega \in [2; 2.373\ldots]$

(iii) Projections / fixed point $\rightsquigarrow$ effective algos. in practice.

Note: Gaussian elimination is also efficient in practice because the boundary matrix M_r usually remains sparse throughout the reduction.
↳ near-linear time in practice.

↳ simple algo. for Gaussian elimination:
reduce M to column-echelon form
up to permutation:



Def: $\mathrm{low}(j) = \begin{cases} \max \{ i \mid M_{ij} \neq 0 \} \\ 0 \text{ if } M_{ij} = 0 \ \forall i \end{cases}$

(row of lowest non-zero entry in column)

↳ reduce M until $\mathrm{low}_{|\{j \mid \mathrm{low}(j) \neq 0\}}$ is injective.

Reduction:

$$M = \begin{bmatrix} c_1 & \cdots & c_{\#K_1} \end{bmatrix}$$

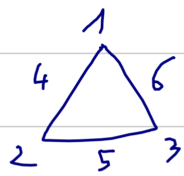
For $j = 1$ to $\#K_1$ do:
  While $\exists\, i < j$ s.t. $\mathrm{low}(i) = \mathrm{low}(j) \neq 0$ do:
    change column content c_j into $c_j + c_i$;

↳ Upon termination: $\mathrm{rank}\, M = \#K_1 - \#\{ j \mid \mathrm{low}(j) = 0 \}$.

Prop: (i) the algo. terminates because every inner loop reduces $\mathrm{low}(j)$ strictly.

(ii) upon termination, $\mathrm{low}_{|\{j \mid \mathrm{low}(j) \neq 0\}}$ is injective by the "while" condition.

Example: Homology of the circle:



$$M_0 \sqcup M_1 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & & & & 1 & & 1 \\ 2 & & \partial_0 = 0 & & \boxed{1} & 1 & \\ 3 & & & & & \boxed{1} & \boxed{1} \\ 4 & & & & & & \\ 5 & & & & & & \\ 6 & & & & & & \end{array}$$

— : low

low is injective until column 5 included

$\Rightarrow$ only column 6 is being reduced:

$c_6 \leftarrow c_6 + c_5$:
$$\begin{array}{c|ccc} & 4 & 5 & 6 \\ \hline 1 & 1 & & 1 \\ 2 & \boxed{1} & 1 & \boxed{1} \\ 3 & & \boxed{1} & \end{array}$$

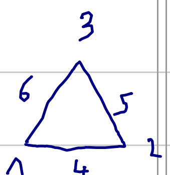
$c_6 \leftarrow c_6 + c_4$:
$$\begin{bmatrix} 1 & & 0 \\ \boxed{1} & 1 & 0 \\ & \boxed{1} & 0 \end{bmatrix}$$

$\Rightarrow \operatorname{rank} M_1 = 2$, $\dim \operatorname{Ker} M_1 = 1$.

↳ interpretation of the matrix reduction:

- scan 1-simplices in an arbitrary order.
- for each one (say σ):
  - either $\partial\sigma = 0$ in the subcomplex spanned by the simplices of the columns to the left $\rightsquigarrow$ σ creates a 1-cycle $\rightsquigarrow$ $\beta_1 + +$
    "homologous"
  - or $\partial\sigma \neq 0$ in that subcomplex $\rightsquigarrow$ the chain $\partial\sigma$ becomes a boundary $\rightsquigarrow$ σ kills an $(k-1)$-cycle $\rightsquigarrow$ $\beta_{k-1} - -$

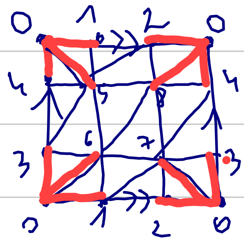
Example: Homology of the circle.



algo:

	1	2	3	4	5	6	
β_0	1	1	1	X	X		$\rightsquigarrow$ 1
β_1						1	$\rightsquigarrow$ 1
β_2	0 throughout (no 2-simplex)						

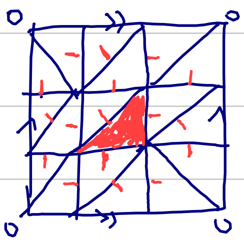
Example: Homology of the torus $S^1 \times S^1$:



pair each red edge with its vertex other than 0 $\longrightarrow$ $\left| \begin{array}{l} \beta_0 = 1 \\ \beta_1 = 0 \end{array} \right.$

(spanning tree)

$\rightarrow$ each remaining edge will create a 1-cycle.



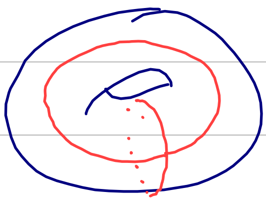
insert $[1,2]$ and $[3,4]$ (β_1++ for each)

then pair each remaining edge with an incident triangle as shown on the left.

$\rightarrow$ $\beta_1 = 2$

$\rightarrow$ the remaining triangle (in red) creates a 2-cycle $\rightarrow \beta_2$++.

In total: $\beta_0 = \beta_2 = 1$, $\beta_1 = 2$.



1 connected component
1 handle + 1 tunnel
1 enclosed void

⑥ Concluding remarks:

It is possible to extend the definitions to:

- arbitrary topological spaces (no need for triangulations)
- other field characteristics (need to orient simplices)
- simplicial maps $K \to L$ or continuous maps $X \to Y$ in such a way that the functorial identities hold.
- arbitrary rings of coefficients (but not useful for us).